SOME ANSWERED AND UNANSWERED QUESTIONS ABOUT THE STRUCTURE OF THE SET OF FERMIONIC ACTIONS WITH GWL SYMMETRY

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1. Introduction

In this talk I will briefly discuss several issues that I thought about since Ginsparg-Wilson (GW) relation and Ginsparg-Wilson-Lüscher (GWL) symmetry became popular topics in lattice field theory. Most of these issues are not resolved to my satisfaction (if at all), which actually makes them an appropriate material to discuss at a workshop like this.

In lattice field theory we typically want to use some finite or countably infinite set of variables to define, as a sequence of approximations, the theory which formally involves a continuous infinity of variables. The most important guide to do this, both correctly and efficiently, are the symmetries. The dynamics of theories relevant in particle physics (such as QCD) is crucially driven by *(i)* Poincaré symmetry *(ii)* gauge symmetry, and *(iii)* chiral symmetries.

Obviously, the lattice counterparts of these do not involve precisely the same transformations, since they act on a different set of degrees of freedom. The goal is rather to choose the discrete set of variables and the set of symmetry conditions so that the dynamics is constrained in a way analogous to that in the continuum. While this is quite non-unique, we usually stick to very definite choices. Thus, trying to account for at least some of the Poincaré invariance, the variables are usually associated with the hypercubic lattice structure, and their Euclidean dynamics is required to respect its symmetries. With gauge invariance in mind, it is most common to associate fermionic variables with sites and gauge group elements with links of the lattice, and to form actions built out of closed gauge loops or

open gauge loops with fermionic variables at the ends. While in this setup it is trivial to restrict the actions further by requiring the invariance under the on-site γ_5 rotation (naive chiral symmetry), for well known reasons, the resulting set of actions is just too small to define the theories we want.

This situation is shown in Fig. 1, where the set A represents acceptable fermionic actions, quadratic in fermionic variables, and with "easy symmetries". The subset A^L of local actions is usually considered (with at least exponentially decaying couplings at large distances in arbitrary gauge background), because of the fear of non-universality in the non-local case. The problems with naive chiral symmetry are reflected by the fact that there is no intersection of the subset of symmetric actions A^C with the subset of doubler-free actions A^{ND} on the local side of the diagram. This is a consequence of the Nielsen-Ninomiya theorem [1].

A possible clean resolution of this is contained in a proposition that lattice theory, for which the chirally nonsymmetric part of the propagator is local, is in virtually all important aspects as good as the one with chirally nonsymmetric part being zero [2, 3]. This is plausible, because the "important aspects" are typically associated with properties of fermionic correlation functions at large distances. These, in turn, depend crucially on the long distance behaviour of the propagator, hence the significance of the above property. The actions satisfying this requirement became known as GW actions, and if we represent the elements of A by corresponding Dirac kernels D, then we have

$$A^{GW} \;\equiv\; \{\, D \in A \,:\, (D^{-1})_N \texttt{ is local} \,\} \qquad (D^{-1})_N \;\equiv\; \frac{1}{2}\gamma_5\{\gamma_5, D^{-1}\}$$

GW kernels are not particularly generic. For free Wilson–Dirac operator in Fourier space we have for example

$$(D_W^{-1})_N \;=\; \frac{\sum_\mu 1-\cos p_\mu}{(\sum_\mu 1-\cos p_\mu)^2+\sum_\mu \sin^2 p_\mu}\,\mathbb{I}$$

where $\mathbb{I}$ is the identity matrix in spinor space. The second partial derivatives of the scalar function in the above expression are directional, implying that the operator is non-local. The chirally nonsymmetric part of the propagator affects the long distance physics, and chiral properties of Wilson–Dirac operator are bad.

The reason why the above considerations are exciting is that it appears that fermion doubling is not a definite property of local GW actions [4, 5, 6]. In other words, $A^{GW} \cap A^{ND} \cap A^L \neq \emptyset$, as indicated on Fig. 1. Apart from a conjectured existence of this intersection, not much is known about the structure of the set A^{GW}. The relevant interesting questions include the following: Are there any useful definite properties of the set A^{GW} and the set

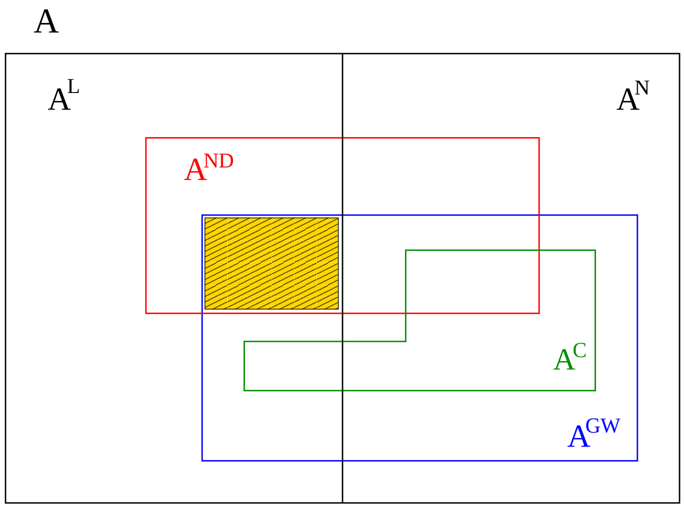

Figure 1. Lattice fermionic actions. Indicated are sets with following properties: A - hypercubic symmetries, gauge invariance, relativistic "naive" continuum limit; A^L - local, A^N - nonlocal, A^{ND} - no doublers, A^C - chiral symmetry, A^{GW} - GWL symmetry.

$A^{GW} \cap A^{ND}$? Can we classify all GW actions by some useful characteristics? How simple can GW actions be? What is a good definition of "simple" for these actions? In what follows, I will discuss certain issues that are relevant to these kind of questions.

2. Non-Ultralocality of GWL Transformations

There is one fully general result here that reveals the inherent property of GW actions and the nature of symmetry they share [8]. Considering the GWL transformations $\delta\psi = i\theta\gamma_5(\mathbb{I} - RD)\psi$, $\delta\overline{\psi} = \overline{\psi}\, i\theta(\mathbb{I} - DR)\gamma_5$, such that $[R, \gamma_5] = 0$, it is well known that the set A^{GW} can be alternatively defined through the symmetry principle [9], i.e.

$$A^{GW} \;\equiv\; \{\, D \in A,\; \exists\, R \;\texttt{local} \;:\; \delta(\overline{\psi} D \psi) = 0 \,\}$$

The following result can be proved [8]:

> *If $D \in A^{GW}$, then the corresponding infinitesimal GWL transformation couples variables at arbitrarily large lattice distances, except when $R = 0$ (standard chiral symmetry).*

This is equivalent to non-ultralocality of $\mathcal{D} \equiv 2RD = 2(D^{-1})_N D$, assigned to any $D \in A^{GW}$, except when $D \in A^C$. Ref. [8] actually deals in detail with the physically relevant case of local elements of A^{GW}, but it is in fact true for all elements.

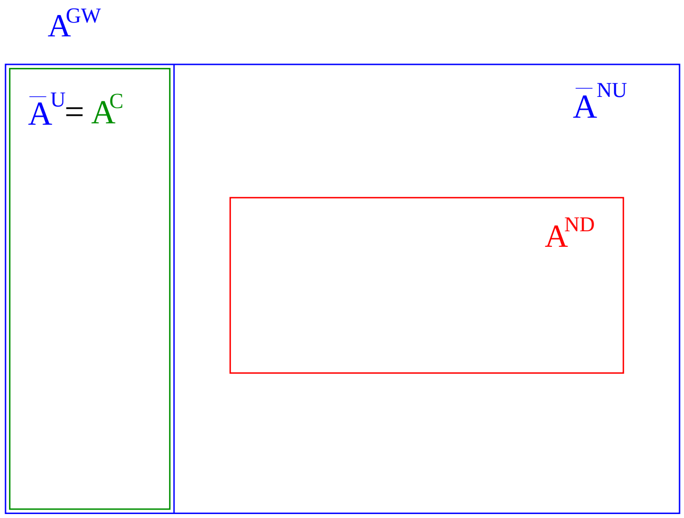

Figure 2. Ultralocality properties of infinitesimal GWL symmetry transformations for the set A^{GW}. Subset $\overline{A}^U$ has ultralocal and $\overline{A}^{NU}$ non-ultralocal transformations.

The above theorem on *"weak non-ultralocality"* is represented on Fig. 2, where the set A^{GW} is split into the parts with ultralocal GWL transformation ($\overline{A}^U$), and non-ultralocal GWL transformation ($\overline{A}^{NU}$). Since $\overline{A}^U = A^C$, this means that there is a sharp discontinuity in the set A^{GW}. Naively, A^C represents a smooth limit in A^{GW} as the chirally nonsymmetric part of the propagator completely vanishes. However, while chiral transformation only mixes variables on a single site, the nontrivial infinitesimal GWL symmetry operation requires rearrangement of infinitely many degrees of freedom (on infinite lattice). This is a necessary requirement to achieve the delicate goal of preserving chiral dynamics, while keeping doublers away.

3. Non-Ultralocality of GW Actions

From the theorem on weak non-ultralocality it follows that, except for the subset A^C, there are no ultralocal elements of A^{GW} for which $R = (D^{-1})_N$ is ultralocal [7, 8]. Apart from conceptual value, this obviously has some serious practical consequences for both perturbation theory and numerical simulations. While the above subset of A^{GW} is the one that is usually considered in the literature, it would be of great interest to know, whether non-ultralocality of GW actions extends to the more general case.

Contrary to the weak non-ultralocality, which holds even in the presence of fermion doubling, non-ultralocality of actions can only hold for doubler–free actions. This is because (at least in free case) there are infinitely many

chirally nonsymmetric ultralocal GW actions with doublers, e.g.

$$D(p) \;=\; \sum_\mu \sin^2 p_\mu \, \mathbb{I} + i \sin p_\mu \, \gamma_\mu \qquad (D^{-1})_N \;=\; \frac{1}{1+\sum_\mu \sin^2 p_\mu} \, \mathbb{I}$$

Consequently, at the free level, the hypothesis of *"strong non-ultralocality"* can be formulated like this [8]

> *HYPOTHESIS: There is no $D(p) \in A$ such that the following three requirements are satisfied simultaneously:*
>
> (α) *$D(p)$ involves finite number of Fourier terms.*
>
> (β) *$(D^{-1}(p))_N$ is analytic.*
>
> (γ) *$(D^{-1}(p))_C$ has no poles except if $p_\mu = 0 \pmod{2\pi}$, $\forall \mu$.*

Conditions ($\alpha - \gamma$) represent ultralocality, GWL symmetry, and the absence of doublers. Below I will describe an algebraic problem which, I believe, holds the key to this issue. My reasoning will necessarily be terse, but the resulting problem will be stated clearly.

If non-ultralocality indeed holds, it will most likely result from the clash of the two analyticity properties $(\beta), (\gamma)$. I will consider the two-dimensional restrictions of the lattice Dirac operators in higher (even) dimensions, because they are already capable of capturing the required analytic structure. As a result of hypercubic symmetry, the restrictions have the form (the term proportional to $\gamma_1\gamma_2$ is ignored for simplicity)

$$D(p) \;=\; A(p)\, \mathbb{I} + i B_\mu(p)\, \gamma_\mu \qquad D^{-1} \;=\; \frac{A\, \mathbb{I} - i B_\mu \gamma_\mu}{A^2 + B_\mu B_\mu}$$

where $p = (p_1, p_2)$, $\mu = 1, 2$ and the functions $A(p), B_\mu(p)$ have the appropriate symmetry properties.

The crucial difference between ultralocal and non-ultralocal actions is that in the former case we only have finite number of coefficients to adjust so that $(\beta), (\gamma)$ are satisfied, while in the latter case there are infinitely many. This is more explicit if one makes the change of variables, such as $x = \sin \frac{p_1}{2}$, $y = \sin \frac{p_2}{2}$, which does not change the analytic structure of the relevant functions. Then we are essentially dealing with polynomials.

It is easy to see that requirement (β) is particularly restrictive because it implies that the symmetric rational function $R(x^2, y^2) \equiv A/(A^2 + B_\mu B_\mu)$ is analytic on the domain $[-1,1] \times [-1,1]$, while the polynomial $A^2 + B_\mu B_\mu$ vanishes at the origin. This is only possible if the numerator and denominator have a common polynomial factor which can be canceled so that the denominator does not vanish anymore. From the structure of $R(x^2, y^2)$ it follows that $A(x^2, y^2)$ and $B(x^2, y^2) \equiv B_\mu B_\mu$ must each have this polynomial factor. It turns out that apart from the necessary zero at the origin,

such common factors $F(x,y)$ tend to possess another zero in the domain $[-1,1]\times[-1,1]$, which then makes the inclusion of requirement (γ) impossible. Consequently, it would be inherently useful to prove or disprove the following hypothesis:

HYPOTHESIS: Let G be the polynomial in x^2, y^2 with complex coefficients, such that $G(0,0)=1$. There is no G such that the polynomial

$$B(x^2,y^2) \;=\; 4x^2(1-x^2)\,G^2(x^2,y^2)+4y^2(1-y^2)\,G^2(y^2,x^2)$$

can be factorized as $B(x^2,y^2) = P(x,y)F(x,y)$, where $P(0,0)\neq 0$, $F(0,0)=0$, and $F(x,y)\neq 0$ elsewhere on the domain $[-1,1]\times[-1,1]$.

The above form of $B(x^2,y^2)$ is dictated by hypercubic symmetries. I stress that if this hypothesis is true, then it implies "strong non-ultralocality" of GW actions. On the other hand, the possible examples of ultralocal GW actions can only be built out of counterexamples to this algebraic statement.

4. Simple GW Actions?

Assuming that GW actions indeed can not be simple in position space (non-ultralocality), one naturally asks what kind of other practically useful properties they *can have*. I propose to examine the possibility that GW actions can be simple in eigenspace.

If the complete left-right eigenset $\{\,|\,\phi_L^i(U)>,\,|\,\phi_R^i(U)>,\lambda_i(U)\,\}$ exists for $D(U)\in A$, then we can represent the operator as

$$D \;=\; \sum_i \,|\,\phi_R^i> \;\lambda_i\; <\phi_L^i\,|$$

This representation is useful even in case of Wilson and staggered fermions, since the effects of light quarks are quickly accounted for by including only the lightest eigenmodes in the sum. The underlying idea is appealing for both generation of dynamical configurations [10, 11], and for propagator technology [12]: once the approximate eigenmode representation is computed, the resulting quark propagators can be tied together in any way desired. The group at the University of Virginia is currently actively pursuing this approach (see also the talk by T. Lippert).

The eigenspace representation of the operator $D(U)$ is "simple", if the corresponding eigenbasis can be calculated efficiently. Even if $D(U)$ is not ultralocal, there still may be a *commuting ultralocal* operator $Q(U)$ with the same eigenbasis. In this case the eigenspace representation of D is as simple as the eigenspace representation of Q. Consequently, it would be very interesting to know whether there are local, doubler-free elements of A^{GW}, for which such an ultralocal operator Q exists.

Obvious candidates for GW actions of the above type would be the functions of the ultralocal operator Q, i.e. $D = F(Q)$. It is an open question whether such GW actions exist. Another simple possibility is to consider the functions $F(Q, Q^+)$, where Q is the ultralocal normal operator $[Q, Q^+] = 0$. The task of finding such actions simplifies a lot if one only considers the operators $Q = D_0$ representing valid, doubler-free lattice Dirac operator with γ_5-hermiticity ($D_0^+ = \gamma_5 D_0 \gamma_5$). This is because to such $D_0 \in A^{ND}$ one can directly assign a doubler-free element $D \in A^{GW}$, in a way analogous to the Neuberger construction [5], i.e.

$$D = m_0 \Big[1 + (D_0 - m_0) \frac{1}{\sqrt{(D_0 - m_0)^+ (D_0 - m_0)}} \Big]$$

with appropriate choice of m_0. One is thus lead to consider the following:

PROBLEM: Are there any ultralocal elements $D_0 \in A^{ND}$ with γ_5-hermiticity that are normal?

This is a beautiful problem with trivial solutions at the free level (e.g. Wilson-Dirac operator), but none are known in arbitrary gauge background.

To get a flavour of what is involved here, it is useful to unmask the spinorial structure of the problem. For example, in two dimensions any operator $D \in A$ has the form $D = A\mathbb{I} + iB_\mu \gamma_\mu + C\gamma_5$ where A, B_μ, C are gauge invariant matrices with position and gauge indices only. γ_5-hermiticity implies that A, B_μ, C are hermitian, and in this case normality demands

$$\{B_\mu, C\} \;=\; i\epsilon_{\mu\nu}[B_\nu, A] \tag{1}$$

The challenge is to find out whether, having only finite number of gauge paths at our disposal, we can arrange for the above identities to hold. This is quite nontrivial and definite properties of A, B_μ, C under hypercubic transformations represent an important constraint here.

I would also like to point out that in the above language, it is easy to understand how γ_5-hermiticity combined with normality simplifies the algebraic structure imposed by GW symmetry. Indeed, if one identifies $J_1 \equiv A - 1, J_2 \equiv B_1, J_3 \equiv B_2$, and $C^2 = 1 - J_\mu J_\mu$, then the canonical GW relation $\{D, \gamma_5\} = D\gamma_5 D$ in two dimensions translates into

$$\{J_\mu, C\} \;=\; i\epsilon_{\mu\nu\rho}[J_\nu, J_\rho] \tag{2}$$

Relations (1) form a subset of the above identities that are automatically satisfied if γ_5-hermiticity and normality are demanded. The algebraic structure (2) implied by GW relation is perhaps interesting by itself and deserves further study.

Finally, I would like to introduce the lattice Dirac operator which might be of practical relevance in the context of using the eigenspace techniques

in lattice QCD. Let $\{\, |\, \phi_L^i(U) >, |\, \phi_R^i(U) >, \lambda_i(U)\, \}$ be the eigenset of the Wilson-Dirac operator $D_W(U)$, and let $m_0 \in (0,2)$. Consider the operator

$$D \;=\; \sum_i \;|\, \phi_R^i > \; m_0 \Big[1 + \frac{\lambda_i - m_0}{\sqrt{(\lambda_i - m_0)^*(\lambda_i - m_0)}} \Big] \; < \phi_L^i \,| \qquad (3)$$

This is a well defined operator for arbitrary gauge background in which the left-right eigenbasis of D_W exists. In trivial gauge background ($U \to 1$) it coincides with the Neuberger operator, and the spectrum allways lies on a circle with radius m_0. While the locality properties of D are questionable, the fact that it is perfectly local in the free limit suggests that non-local parts, which might be present, will be arbitrarily small on sufficiently smooth backgrounds. Even though this can certainly cause practical concerns at intermediate couplings, it would seem unlikely that there is a problem of principle as the continuum limit is approached.

Operator (3) should have improved chiral properties relative to the Wilson-Dirac operator, while its computational demands in the eigenspace approach are approximately the same. The degree to which the chirally non-symmetric part of the propagator is local in nontrivial backgrounds (it is proportional to a delta function in free limit) is an open question. At the same time, however, the fact that the spectrum is forced on a circle suggests that the additive mass renormalization will be small (if any). These issues are currently under investigation.

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References

1. H. B. Nielsen and M. Ninomiya, Phys. Lett. **B105** (1981) 219; Nucl. Phys. **B185** (1981) 20[E: B195 (1982) 541].
2. P. Ginsparg and K. Wilson, Phys. Rev. **D25** (1982) 2649.
3. P. Hasenfratz, Nucl. Phys. **B525** (1998) 401.
4. P. Hasenfratz, Nucl. Phys. **B** (Proc. Suppl.) **63** (1998) 53.
5. H. Neuberger, Phys. Lett **B417** (1998) 141.
6. P. Hernández, K. Jansen and M. Lüscher, Nucl.Phys. **B552** (1999) 363.
7. I. Horváth, Phys. Rev. Lett. **81** (1998) 4063.
8. I. Horváth, Phys. Rev. **D60** (1999) 034510.
9. M. Lüscher, Phys. Lett. **B428** (1998) 342.
10. A. Duncan et al, Phys. Rev. **D59** (1999) 014505.
11. I. Horváth, `hep-lat/9909044`.
12. T.L. Ivanenko and J. Negele, Nucl. Phys. **B** (Proc. Suppl.) **63** (1998) 504.